Hausdorff towers and gaps

Piotr Borodulin-Nadzieja

Winterschool 2013, Hejnice

joint work with David Chodounsky

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Hausdorff towers and gaps

Motiv	vation	Hausdorff towers	Suslin towers	Special and Hausdorff C	aps
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	Tower				
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	$ T_{\alpha} \subseteq \omega$	for each α ;			
	$\blacksquare \ T_{\alpha} \setminus T_{\beta}$	$_{3}$ is finite ($\mathcal{T}_{lpha}\subseteq^{*}$	T_{β}) iff $\alpha < \beta$.		

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Question 1

Is there a tower $(T_{\alpha})_{\alpha}$ such that $T_{\alpha} \nsubseteq T_{\beta}$ for each $\alpha < \beta$?

Question 1 reformulated

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Is there an uncountable family \mathcal{T} \subseteq [\omega]^{\omega} such that

(\mathcal{T}, \subseteq^*) is well-ordered;
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$$(\mathcal{T},\subseteq)$$
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A family $(L_{\alpha}, R_{\alpha})_{\alpha < \omega_1}$ is a *gap* if

- (L_{α}) $_{\alpha}$ and $(R_{\alpha})_{\alpha}$ are towers;
- $L_{\alpha} \cap R_{\alpha} = \emptyset$ for each α ;
- there is no $L \subseteq \omega$ s.t. $L_{\alpha} \subseteq^* L$ and $R_{\alpha} \cap L =^* \emptyset$ for each α .

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Answer to Question 1 (S. Todorčević).

Let $(L_{\alpha}, R_{\alpha})_{\alpha}$ be a Hausdorff gap. Define $f(dom(f) = \omega_1)$:

$$f(\alpha) = \{\beta < \alpha \colon L_{\beta} \cap R_{\alpha} = \emptyset\}.$$

Hausdorff condition $\implies f: \omega_1 \to [\omega_1]^{<\omega};$

f is a set-mapping (i.e. $\alpha \notin f(\alpha)$);

Free-set theorem \implies there is $\Lambda \subseteq \omega_1$, $|\Lambda| = \omega_1$ such that

 $\alpha \notin f(\beta)$ for each $\alpha, \beta \in \Lambda$;

If
$$L_{\beta} \cap R_{\alpha} \neq \emptyset$$
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Motivation		Special and Hausdorff Gaps
Answer.		

pre-Definition: Hausdorff tower

A tower $(T_{\alpha})_{\alpha}$ is Hausdorff if (it contains a subtower (T'_{α}) such that) for each α and n

$$\{\beta < \alpha : T_{\beta} \setminus T_{\alpha} \subseteq n\}$$
 is finite.

Proposition

There is a Hausdorff tower. Each Hausdorff tower contains an uncountable *G*-antichain.

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Is there a tower without an uncountable \subseteq -antichain?

Theorem (Kunen, van Douwen, 1982)

 $CH \implies Yes.$

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Basic definitions.

Definition: Hausdorff tower

A tower $(T_{\alpha})_{\alpha}$ is *Hausdorff* if (it contains a subtower (T'_{α}) such that) for each α and n

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Definition: Suslin tower

A tower $(T_{\alpha})_{\alpha}$ is *Suslin* if it does not contain an uncountable \subseteq -antichain.

Definition: special tower

A tower $(T_{\alpha})_{\alpha}$ is *special* if it is not Suslin.

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Hausdorff towers and gaps

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Hausdorff towers and gaps

Martin's Axiom (+ non-CH).

Proposition

 $MA(\omega_1) \implies$ each tower is Hausdorff.

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 $MA(\omega_1) \implies$ for each tower $(T_{\alpha})_{\alpha}$ there is a tower $(T'_{\alpha})_{\alpha}$ such that

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OCA and PID.

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 $OCA \implies$ each tower is special.

Theorem

Assume PID. Then each tower is Hausdorff if and only if $\mathfrak{b} > \omega_1$.

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There exists a Suslin tower of size \mathfrak{b} . Consistently, there is a Suslin tower of size less than \mathfrak{b} .

Theorem (Todorčević)

If a tower generates a non-meager ideal, then it is Suslin.

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Hausdorff towers and gaps

Suslin tower from a Cohen real.

Proposition

Let \mathbb{C} be the Cohen forcing and let $(T_{\alpha})_{\alpha}$ be a tower. Then

 $\Vdash_{\mathbb{C}} (\dot{c} \cap T_{\alpha})_{\alpha} \text{ is a Suslin tower.}$

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Hausdorff towers and gaps

$\blacksquare \Vdash_{\mathbb{C}} (\dot{c} \cap \mathcal{T}_{\alpha})_{\alpha} \text{ is a tower;}$

- Fix $p \in 2^n$;
- Let $\Vdash_{\mathbb{C}} \dot{X} \subseteq \omega_1$;
- WLOG, $X \in V$;
- Let $\alpha < \beta \in X$ such that $T_{\alpha} \cap n = T_{\beta} \cap n$;
- Let *m* be such that $T_{\alpha} \setminus T_{\beta} \subseteq m$;
- Let q|n = p and q(i) = 0 for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_{\alpha} \subseteq \dot{c} \cap T_{\beta}.$

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Suslin tower from a Cohen real - proof.

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- Let q|n = p and q(i) = 0 for each $i \in [n, m)$;
- $q \Vdash \dot{c} \cap T_{\alpha} \subseteq \dot{c} \cap T_{\beta}.$

		Special and Hausdorff Gaps
Special m	200	

A gap $(L_{\alpha}, R_{\alpha})_{\alpha}$ is *special* if there is an uncountable $X \subseteq \omega_1$ such that

$$(L_{\alpha} \cap R_{\beta}) \cup (L_{\beta} \cap R_{\alpha}) \neq \emptyset$$

for each $\alpha < \beta \in X$.

Oriented gap

A gap $(L_lpha,R_lpha)_lpha$ is *oriented* if there is an uncountable $X\subseteq \omega_1$ such that

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Basic facts

notation: "Hausdorff" = "equivalent to Hausdorff";

- Hausdorff \implies oriented \implies special;
- If $(L_{\alpha}, R_{\alpha})_{\alpha}$ is Hausdorff, then $(L_{\alpha})_{\alpha}$ is Hausdorff;
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Questions - Scheepers (1993)

Is every oriented gap Hausdorff? Is every special gap oriented?

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Hausdorff towers and gaps

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Consistently, there is an oriented but non-Hausdorff gap.

Theorem

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• Let $(T_{\alpha})_{\alpha}$ be a Suslin tower.

(Spasojević, 1996) There is a *σ*-centered forcing ℙ and a ℙ-name (*L*_α)_α for a tower such that

 $\Vdash_{\mathbb{P}} (\dot{L}_{\alpha}, \mathcal{T}_{\alpha})_{\alpha}$ is an oriented gap.

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- $\Vdash_{\mathbb{P}}$ " $(T_{\alpha})_{\alpha}$ is Suslin", since σ -centered forcings cannot destroy ccc.
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 $\Vdash_{\mathbb{P}} (\dot{L}_{\alpha})_{\alpha}$ is not a Hausdorff tower.

Therefore

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Hausdorff towers and gaps

Special non-Hausdorff tower.

Corollary

Consistently, there is a special tower which is not Hausdorff.

Theorem

If $(T_{\alpha})_{\alpha}$, $(T'_{\alpha})_{\alpha}$ generates the same ideal, and $(T_{\alpha})_{\alpha}$ is Hausdorff, then $(T'_{\alpha})_{\alpha}$ is Hausdorff.

 Consistently, there is a special tower and a Suslin tower generating the same ideal.

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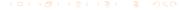
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Special and Hausdorff Gaps

Acknowledgements: INFTY Network.





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Acknowledgements: Fields Institute.

Figure: CN tower



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